Exam sets October 2022

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All items count equally so make sure you try all exercises. Good luck!

1. Consider the sets $X = \{1, 2, 3, 4, 5\}$, $Y = \{2, 4, 6, 8\}$ and $Z = \{8, -1, 4\}$.

   (a) Write down the set $(X \cap Y) \setminus Z$ by listing all its elements explicitly.

   (b) Explicitly define a function $f : Z \to X$ that is not injective.

   (c) Define the function $g : Y \to Z$ by $g(2) = g(4) = 4$ and $g(6) = -1$ and $g(8) = 8$. Write down the set $g^{-1}(\{4, -1\})$ by listing all its elements explicitly.

2. In this exercise X and Y denote two arbitrary sets.

   (a) Prove that $2^X \cap 2^Y \subseteq 2^{X \cap Y}$.

   (b) Prove that if $X \subseteq Y$ and we have an equivalence relation R on Y then $E = X^2 \cap R$ is an equivalence relation on X.

   (c) Define a bijection $b : X \times Y \to Y \times X$ and prove that the b you constructed is indeed a bijection.

3. (a) Give an example of an infinite set that is NOT countably infinite.

   (b) Prove that for any non-empty set X and any $x \in X$ we have $\{x\} \cup (X \setminus \{x\}) = X$.

   (c) Prove by induction that $\prod_{k=0}^{n} 4^k = 2^{n(n+1)}$ is true for any $n \in \mathbb{N}$.

4. Introduce the sets $X = \{a, b, c\}$ and $Y = \{4, 5\}$ and $Z = \{X, Y\}$. On these sets we have the functions $f : X \to Y$ and $g : Y \to Z$ defined by $f(a) = 4$ and $f(b) = f(c) = 5$ and $g(4) = Y$ and $g(5) = X$.

   (a) Explain why g has an inverse and compute $g^{-1}(X)$ and $g^{-1}(Y)$.

   (b) Is the composition $g \circ f$ surjective?

   (c) Give a system of representants for the equivalence relation $\sim_f$ on X.